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ORIGINAL ARTICLE

Solving Matrix Games with I-fuzzy Payoffs: Pareto-optimal Security Strategies Approach

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Abstract A recent research work of Clemente et al. [12] on Pareto-optimal security strategies (POSS) in matrix games with fuzzy payoffs is extended to I-fuzzy scenario. Besides, the membership and the non-membership functions of the I-fuzzy values for both players are obtained by employing the technique of multiobjective optimization. The presented approach provides an efficient solution to a class of I-fuzzy matrix games with piecewise linear membership and non-membership functions. This class also includes I-fuzzy matrix games with triangular and trapezoidal I-fuzzy numbers as special cases. Further, POSS approach also provides an approximate solution to I-fuzzy matrix games with payoffs as general I-fuzzy numbers.

Keywords Fuzzy matrix game · Intuitionistic fuzzy sets · I-fuzzy payoffs · Pareto-optimal security strategies

1. Introduction

Atanassov [3] integrated the notion of hesitancy degree in the definition of fuzzy set by adding a new component which describes the degree of non-membership in

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the fuzzy set; thereby he extended and introduced a new concept of what he called intuitionistic fuzzy set. While the fuzzy set defines the degree of membership of an element to a given set and its non-membership degree is understood as one minus its membership degree, the intuitionistic fuzzy set provides more-or-less independent degree of membership and degree of non-membership of an element in a given set. The only requirement in the latter is that the sum of the two degrees is not greater than 1. As a result, an intuitionistic fuzzy set exhibits characteristics of affirmation, negation and hesitation. For instance, in any confronting situation in decision making, besides support or positive response and objection or negative response, there could be an abstention indicating hesitation and indeterminacy in response to the situation. Intutionistic fuzzy sets occur very naturally in real life decision making problems. There are numerous applications reported in literature in a variety of areas; for more on it we may refer to Atanasov [5], Szmidt and Kacprzyk [28], and references cited therein.

There are certain controversies (see, Dubois et al. [13] and Grzegorzewski and MrÓwka [17]) regarding the nomenclature of Atanassov's intuitionistic fuzzy set because similar nomenclature has been also used in intuitionistic logic, and the two completely differ in their mathematical structure and treatment. It obviously makes sense to avoid using same terminology for two different concepts. For this reason, as suggested in [13] and [17], henceforth Atanassov's intuitionistic fuzzy set is called I-fuzzy set.

Although the I-fuzzy sets posses rich structure yet there are only few good studies to report, for instance [6, 23, 24], that utilize the I-fuzzy sets in matrix games. Moreover, these studies are relatively recent ones and the applications of I-fuzzy sets, especially in optimization, is not yet fully explored. On the other hand, fuzzy linear programming problems and fuzzy matrix games have been extensively studied in literature, for instance see, Bector and Chandra [8], Nischzaki and Sakawa [26], and numerous references cited therein. The earliest study of a two person zero sum matrix game (TPZSMG) with fuzzy payoffs is due to Campos [11]. Later Bector et al. [9] interpreted Campos's model in context of fuzzy linear programming duality and showed that solving a TPZSMG with fuzzy payoffs is equivalent to solving an appropriate pair of primal-dual fuzzy linear programming problems [7]. Both these studies employed Yager's ranking function [30] for defuzzification purpose. But the drawback of these studies is that they do not provide membership functions of the optimal fuzzy values of two players in a TPZSMG with fuzzy payoffs. On the other hand, related research papers of Li [18, 19] provide the membership function of the fuzzy value but his study is restricted to games with payoffs prescribed by triangular fuzzy numbers (TFNs) only.

Recently, Aggarwal et al. [1, 2] extended the study of Bector et al. [8-10] and Vijay et al. [29] to I-fuzzy scenario. Nan and Li [23] computed the average index of the I-fuzzy value to study TPZSMG in which payoff matrix entries are triangular I-fuzzy numbers (TIFNs). Clemente et al. [12] studied TPZSMG with payoff matrix prescribed by general fuzzy numbers not necessarily restricted to any particular family of fuzzy numbers.

It is well known that a fuzzy number can be characterized completely by its α-cuts.

Also, most of the fuzzy numbers arising in practical situations have piecewise linear membership functions, and it takes only finitely many α-cuts for approximating such fuzzy numbers. Clemente et al. [12] advocated this argument by applying standard fuzzy order due to González and Vila [15, 16] in fuzzy matrix game. This allowed Clemente et al. [12] to provide complete description of the fuzzy value of game for each of the two players. Thus the approach in [12] seems to be superior to all well known methods [9, 11, 18] for solving matrix games with fuzzy payoffs.

Now, since an I-fuzzy number can be described in terms of two fuzzy numbers one with membership function and the other one with one minus the non-membership function, therefore it is possible to characterize the I-fuzzy number completely in terms of it α-cuts. This motivates us to see how the approach of Clemente et al. [12] can be adopted to I-fuzzy scenario. It may be added that our approach will provide an efficient methodology to I-fuzzy matrix games with piecewise linear membership and non-membership functions. The study naturally includes the class of I-fuzzy matrix games with triangular I-fuzzy numbers and trapezoidal I-fuzzy numbers. We will also create a POSS for each player and give an I-fuzzy interpretation of the payoff value for each player. The POSS approach also provides insights to generate approximate solutions to those I-fuzzy matrix games whose payoffs are given by general I-fuzzy numbers.

The present study has certain obvious advantages over the related studies on same problem of TPZSMG with I-fuzzy payoffs. Clemente et al. [12] generated only one efficient point (or solution of game) whereas the method described by us can generate many more efficient points thereby giving a fair idea of efficient frontier for I-fuzzy values of two players in an I-fuzzy game. The two players (here the decision makers) are offered the opportunity to choose their respective efficient points according to their own preferences. For example, if the payoffs are triangular I-fuzzy numbers, a player may give more importance to the modal value compared to the average value. The components of the chosen efficient point (or vector) will finally yield the approximate membership and non-membership functions of the I-fuzzy values of the I-fuzzy matrix game under consideration. Our work also differs from that of Nan and Li [23] in the way that we are able to characterize the membership functions and the non-membership functions of the optimal I-fuzzy values of two players rather than their average indexes, which are merely crisp (defuzzified) numbers. We observe that once we have a full description of membership and nonmembership functions of I-fuzzy values of the game we can easily compute the average indexes for two players but the converse is obviously not true.

The remaining paper is structured as follows. Section 2 presents the preliminaries of I-fuzzy sets and I-fuzzy numbers. Section 3 defines the ranking method for I-fuzzy numbers which is extension of the standard fuzzy order of González and Vila [15, 16]. Section 4 develops a methodology to solve TPZSMGs with I-fuzzy payoffs and provides POSS for each player. Section 5 includes a brief description of the solution methodology, its implementation issues, followed by a numerical illustration. Some conclusions are drawn in Section 6 and Appendix consisting of another detailed example is included at the end.

2. Preliminaries

In this section, we present certain preliminaries with regard to I-fuzzy sets and I-fuzzy numbers. For results relating to fuzzy sets and fuzzy numbers we can refer to Bector and Chandra [8] and Zimmermann [31]. Further, we shall be mainly following the notations of Clemente et al. [12] in the sequel.

Let X be a universal set.

Definition 2.1 (I-fuzzy set) [5] *An I-fuzzy set $\tilde{a}$ in X is described by*

$$\tilde{a} = \{\langle x, \mu_{\tilde{a}}(x), \nu_{\tilde{a}}(x)\rangle \mid x \in X\},$$

where $\mu_{\tilde{a}} : X \to [0,1]$ and $\nu_{\tilde{a}} : X \to [0,1]$ define, respectively, the membership function and the non-membership function.

In other words, $\mu_{\tilde{a}}(x)$ and $\nu_{\tilde{a}}(x)$ are respectively membership degree and non-membership degree of an element $x \in X$ to the set $\tilde{a}$ with $0 \leqq \mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) \leqq 1$. If $\mu_{\tilde{a}}(x) + \nu_{\tilde{a}}(x) = 1$ for all $x \in X$, then $\tilde{a}$ degenerates to the standard fuzzy set.

We now take $X = \mathbb{R}$, the real Euclidean space, and recall an I-fuzzy number.

Definition 2.2 (I-fuzzy number) [25] *An I-fuzzy number $\tilde{a}$ is an I-fuzzy set over $\mathbb{R}$ whose membership function $\mu_{\tilde{a}} : \mathbb{R} \to [0,1]$ and $\nu_{\tilde{a}} : \mathbb{R} \to [0,1]$ satisfy the following conditions:*

(i) *there are real numbers c and d such that $\mu_{\tilde{a}}(c) = 1$ and $\nu_{\tilde{a}}(d) = 1$;*

(ii) *$\mu_{\tilde{a}}$ is quasi concave and $\nu_{\tilde{a}}$ is quasi convex on $\mathbb{R}$;*

(iii) *$\mu_{\tilde{a}}$ is upper semi-continuous and $\nu_{\tilde{a}}$ is lower semi-continuous;*

(iv) *the support sets $\{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) > 0\}$ and $\{x \in \mathbb{R} \mid \nu_{\tilde{a}}(x) < 1\}$ are bounded.*

We denote the set of I-fuzzy numbers by $IFN(\mathbb{R})$.

From the above definition we get at once that for any I-fuzzy number $\tilde{a}$ there exist eight numbers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}$ such that $b_1 \leqq a_1 \leqq b_2 \leqq a_2 \leqq a_3 \leqq b_3 \leqq a_4 \leqq b_4$ and four functions $f_1, f_2, f_3, f_4 : \mathbb{R} \to [0,1]$, called the sides of an I-fuzzy number, where f_1 and f_4 are non-decreasing and f_2 and f_3 are non increasing functions. The membership function $\mu_{\tilde{a}}$ of an I-fuzzy number $\tilde{a}$ can be specified as

$$\mu_{\tilde{a}}(x) = \begin{cases} 0\,, & x < a_1\,, \\ f_1(x)\,, & a_1 \leqq x < a_2\,, \\ 1\,, & a_2 \leqq x \leqq a_3\,, \\ f_2(x)\,, & a_3 < x \leqq a_4\,, \\ 0\,, & x > a_4\,, \end{cases}$$

while the nonmembership function $\nu_{\tilde{a}}$ has the following form

$$\nu_{\tilde{a}}(x) = \begin{cases} 1\,, & x < b_1\,, \\ f_3(x)\,, & b_1 \leqq x < b_2\,, \\ 0\,, & b_2 \leqq x \leqq b_3\,, \\ f_4(x)\,, & b_3 < x \leqq b_4\,, \\ 1\,, & x > b_4. \end{cases}$$

It is worth noting that each I-fuzzy number $\tilde{a}$ is a conjunction of two fuzzy numbers, the membership function of one is $\mu_{\tilde{a}}$ and that of the other is $1 - \nu_{\tilde{a}}$.

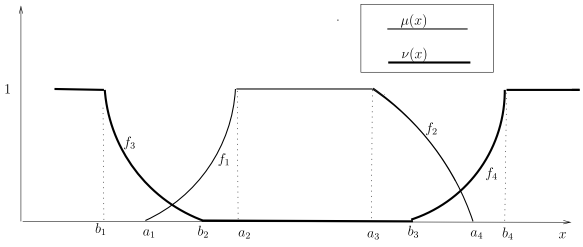

Fig. 1 Membership and non membership functions

We can represent a triangular I-fuzzy number (TIFN) $\tilde{a} = [[a, d_1, d_2], [b, d_3, d_4]]$, where $a_1 = a - d_1$, $a_2 = a_3 = a$, $a_4 = a + d_2$, $b_1 = b - d_3$, $b_2 = b_3 = b$, $b_4 = b + d_4$. Note that $a = b$, $d_3 \geqq d_1 \geqq 0$, $d_2 \geqq d_4 \geqq 0$. Similarly, a trapezoidal I-fuzzy number (TrIFN) as $\tilde{a} = [[a_2, d_1; a_3, d_2], [b_2, d_3; b_3, d_4]]$, where $a_1 = a_2 - d_1$, $a_4 = a_3 + d_2$, $b_1 = b_2 - d_3$, $b_4 = b_3 + d_4$. Note that $d_s \geqq 0$, $s = 1, \cdots, 4$.

It is well known that the α-cuts are the most useful tool for studying fuzzy numbers. We expect a similar situation with I-fuzzy numbers as well, and therefore define the analogous notion for I-fuzzy number.

Definition 2.3 ((α,β)-cut of an I-fuzzy number) [25] *Let $\tilde{a} \in IFN(\mathbb{R})$, and $(\alpha,\beta) \in (0,1]^2$. Then the pair of crisp intervals $a^{\mu}_{\alpha} = \{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) \geqq \alpha\}$ and $a^{\nu}_{\beta} = \{x \in \mathbb{R} \mid \nu_{\tilde{a}}(x) \leqq 1 - \beta\}$ is called the (α,β)-cut of $\tilde{a}$.*

For $\alpha = 0$, we have $a^{\mu}_{0} = cl\{x \in \mathbb{R} \mid \mu_{\tilde{a}}(x) > 0\}$, where cl denotes the closure of the set. Similarly, for $\beta = 0$, we have $a^{\nu}_{0} = cl\{x \in \mathbb{R} \mid \nu_{\tilde{a}}(x) < 1\}$. Since (α,β)-cuts of an I-fuzzy number $\tilde{a}$ are closed intervals, we have

$$a^{\mu}_{\alpha} = [(a^1_{\alpha})^{\mu}, (a^2_{\alpha})^{\mu}]\,, \qquad a^{\nu}_{\beta} = [(a^1_{\beta})^{\nu}, (a^2_{\beta})^{\nu}]\,,$$

where

$$(a^1_{\alpha})^{\mu} = \inf(a^{\mu}_{\alpha})\,, \quad (a^2_{\alpha})^{\mu} = \sup(a^{\mu}_{\alpha})\,, \quad (a^1_{\beta})^{\nu} = \inf(a^{\nu}_{\beta})\,, \quad (a^2_{\beta})^{\nu} = \sup(a^{\nu}_{\beta}).$$

3. Ranking I-fuzzy Numbers

Though there are various methods proposed in literature to compare and rank fuzzy numbers, we describe below the one due to González and Vila [15, 16] that had been used by Clemente et al. [12] in their study.

Let $FN(\mathbb{R})$ be the set of all fuzzy numbers and $\tilde{a} \in FN(\mathbb{R})$.

Definition 3.1 (Standard fuzzy ranking function) *Let $\gamma_{\mu} = \{\alpha_1, \cdots, \alpha_r\} \subseteq [0,1]$ with $\alpha_1 < \cdots < \alpha_r = 1$. A standard fuzzy ranking function is a function $f : FN(\mathbb{R}) \to \mathbb{R}^{2r}$,* *defined as*

$$f(\tilde{\mathfrak{a}}) = (p_{ij}(\tilde{\mathfrak{a}})) = \left(a^j_{\alpha_i}, \quad (i = 1, \cdots, r, \ j = 1, 2)\right).$$

Here, the notation $p_{ij}(\tilde{\mathfrak{a}}) = a^j_{\alpha_i}$ stands for the lower bound ($j = 1$) and the upper bounds ($j = 2$) of an α_i ($i = 1, \cdots, r$)-cut of a fuzzy number $\tilde{\mathfrak{a}}$.

Remark 3.1 Although $f(\tilde{\mathfrak{a}}) \in \mathbb{R}^{2r}$ yet if the numeric values at the end points are same in the α-cuts computation, then we write only the distinct ones in the mathematical expression of $f(\tilde{\mathfrak{a}})$. Consequently, we may have, $f(\tilde{\mathfrak{a}}) \in \mathbb{R}^{2r'}$ with $r' \leq r$, but we shall continue to use $f(\tilde{\mathfrak{a}}) \in \mathbb{R}^{2r}$ in the theoretical development.

Using this function, any partial order on $\mathbb{R}^{2r}$ induces a natural ordering on $FN(\mathbb{R})$. Clemente et al. [12] took the componentwise partial order on $\mathbb{R}^{2r}$. They also called the function f a standard ranking function. We shall be calling it a *standard fuzzy ranking function* to distinguish it from the analogous concept for I-fuzzy scenario that we shall be introducing shortly.

Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^n_+$ is its non-negative orthant.

Definition 3.2 (Standard fuzzy order) [12] *An order $\lessapprox^F$ is called a standard fuzzy order if for $\tilde{\mathfrak{a}}, \tilde{\mathfrak{b}} \in FN(\mathbb{R})$,*

$$\tilde{\mathfrak{a}} \lessapprox^F \tilde{\mathfrak{b}} \Leftrightarrow f(\tilde{\mathfrak{a}}) \leqq f(\tilde{\mathfrak{b}}).$$

Here $\leqq$ in the second inequality stands for componentwise ordering in $\mathbb{R}^{2r}$.

We also understand $\tilde{\mathfrak{a}} \gtrapprox^F \tilde{\mathfrak{b}}$ in terms of $-\tilde{\mathfrak{a}} \lessapprox^F -\tilde{\mathfrak{b}}$. The ordering $\lessapprox^F$ is to be understood in the sense of equivalence classes only because the standard fuzzy order defined by standard fuzzy ranking function f induces the following equivalence relation on $FN(\mathbb{R})$

$$\tilde{\mathfrak{a}} \approxeq^F \tilde{\mathfrak{b}} \Leftrightarrow f(\tilde{\mathfrak{a}}) = f(\tilde{\mathfrak{b}}).$$

This equivalence relation is called an *indifference relation*. Further we use notation $\tilde{\mathfrak{a}} \precsim^F \tilde{\mathfrak{b}}$ to mean $f(\tilde{\mathfrak{a}}) \leqq f(\tilde{\mathfrak{b}})$ and $f(\tilde{\mathfrak{a}}) \neq f(\tilde{\mathfrak{b}})$.

Let $\tilde{\mathfrak{a}} = [a, d_1, d_2]$ and $\tilde{\mathfrak{b}} = [b, u_1, u_2]$ be TFNs. Let $\gamma_\mu = \{\alpha_1 = 0, \alpha_2 = 1\}$. Then, $f(\tilde{\mathfrak{a}}) = (a^1_1, a^2_1, a^1_2)$ and $f(\tilde{\mathfrak{b}}) = (b^1_1, b^2_1, b^1_2)$. It is important to note that for TFNs, both end points for $\alpha_2 = 1$-cut coincide thus $a^2_2 = a^1_2$ and $b^2_2 = b^1_2$. Consequently, in the expressions of $f(\tilde{\mathfrak{a}})$ and $f(\tilde{\mathfrak{b}})$, we avoid writing the same number twice and hence $f(\tilde{\mathfrak{a}}), f(\tilde{\mathfrak{b}}) \in \mathbb{R}^3$ instead of $\mathbb{R}^4$. So, $\tilde{\mathfrak{a}} \lessapprox^F \tilde{\mathfrak{b}}$ if $a^j_1 \leqq b^j_1, \ j = 1, 2$ and $a^1_2 \leqq b^1_2$. For example, consider $\tilde{\mathfrak{a}} = [172, 12, 5]$ and $\tilde{\mathfrak{b}} = [175, 5, 5]$. Then, $a^1_1 = 172 - 12 = 160$, $a^2_1 = 172 + 5 = 177$, are the end-points of the $\alpha_1 = 0$-cut of $\tilde{\mathfrak{a}}$ and $a^1_2 = 172 = a^2_2$, the $\alpha_1 = 1$-cut of $\tilde{\mathfrak{a}}$. Similarly, $b^1_1 = 170$, $b^2_1 = 180$, $b^1_2 = 175 = b^2_2$. Hence, $f(\tilde{\mathfrak{a}}) = (160, 177, 172)$ and $f(\tilde{\mathfrak{b}}) = (170, 180, 175)$. Clearly, $\tilde{\mathfrak{a}} \precsim^F \tilde{\mathfrak{b}}$, and hence $\tilde{\mathfrak{a}} \lessapprox^F \tilde{\mathfrak{b}}$.

As a consequence of Remark 3.1, in the numeric illustrations of vector $f(\tilde{a})$, we shall be using the distinct components only. Thus, we may have $f(\tilde{\mathfrak{a}}) \in \mathbb{R}^{2r'}$ with

$r^{'} \leq r$, for r numbers of α-cuts. However, theoretically we shall continue to write vector $f(\tilde{a}) \in \mathbb{R}^{2r}$.

Taking motivation from the above notions, we describe below a procedure for ranking I-fuzzy numbers.

Definition 3.3 (Standard I-fuzzy ranking function) *Let $\gamma_\mu = \{\alpha_1, \cdots, \alpha_r\} \subseteq [0,1]$ and $\gamma_\nu = \{\beta_1, \cdots, \beta_s\} \subseteq [0,1]$ with $\alpha_1 < \cdots < \alpha_r = 1$ and $1 = \beta_1 > \cdots > \beta_s$. A standard I-fuzzy ranking function is a function $\Phi : IFN(\mathbb{R}) \to \mathbb{R}^{2r+2s}$ given by $\Phi(\tilde{a}) = (g(\tilde{a}), h(\tilde{a}))$, where*

$$g(\tilde{a}) = (p^{\mu}_{ij}(\tilde{a})) = \left((a^{j}_{\alpha_i})^{\mu},\ \ (i = 1, \cdots, r,\ j = 1, 2)\right) \in \mathbb{R}^{2r},$$
$$h(\tilde{a}) = (q^{\nu}_{tj}(\tilde{a})) = \left((a^{j}_{\beta_t})^{\nu},\ \ (t = 1, \cdots, s,\ j = 1, 2)\right) \in \mathbb{R}^{2s}.$$

Here $(a^{j}_{\alpha_i})^{\mu}$ and $(a^{u}_{\beta_t})^{\nu}$ are defined as in Section 2.

Remark 3.2 We can extend the noting in Remark 3.1 for $g(\tilde{a})$ and $h(\tilde{a})$, and hence for $\Phi(\tilde{a})$.

Definition 3.4 (Standard I-fuzzy order) *An order $\lessapprox^{IF}$ is called the standard I-fuzzy order if for $\tilde{a}, \tilde{b} \in IFN(\mathbb{R})$,*

$$\tilde{a} \lessapprox^{IF} \tilde{b} \Leftrightarrow \Phi(\tilde{a}) \leqq \Phi(\tilde{b}) \quad \text{or} \quad \Phi(\tilde{b}) - \Phi(\tilde{a}) \in \mathbb{R}^{2r+2s}_{+}.$$

We continue to use $\leqq$ for the component wise ordering in $\mathbb{R}^{2r+2s}$.

The expression $\tilde{a} \gtrapprox^{IF} \tilde{b}$ is to be understood analogously as $-\tilde{a} \lessapprox^{IF} -\tilde{b}$. We would like to specify that if $\mu_{\tilde{a}}$ and $\nu_{\tilde{a}}$ are respectively the membership and the non-membership function of the I-fuzzy number $\tilde{a}$, then $\nu_{\tilde{a}}$ and $\mu_{\tilde{a}}$ are respectively the membership and the non-membership functions of the I-fuzzy number $-\tilde{a}$, the negative of the I-fuzzy number $\tilde{a}$. We shall be taking this convention in our subsequent discussion.

For instance, if $\tilde{a} = [[\dot{a}, d_1, d_2], [\hat{a}, d_3, d_4]]$ is a TIFN, then by Definition 2.2, $\hat{a} - d_3 \leqq \dot{a} - d_1 \leqq \hat{a} \leqq \dot{a} \leqq \hat{a} \leqq \dot{a} + d_2 \leqq \hat{a} + d_4$, implying $0 \leqq d_1 \leqq d_3,\ 0 \leqq d_2 \leqq d_4,\ \dot{a} = \hat{a}$. Let $\gamma_\mu = \{\alpha_1 = 0,\ \alpha_2 = 1\}$ and $\gamma_\nu = \{\beta_1 = 1,\ \beta_2 = 0\}$. Then $\Phi(\tilde{a}) = (\dot{a}-d_1, \dot{a}+d_2, \dot{a}, \dot{a}-d_3, \dot{a}+d_4, \dot{a}, \dot{a})$. If $\tilde{a} = [(175, 5, 5), (175, 8, 10)]$, then $\Phi(\tilde{a}) = (170, 180, 175, 167, 185)$. Note that, on account of Remark 3.2, while computing the end points of the α_i-cuts and β_t-cuts, only the distinct numeric values are written for $\Phi(\tilde{a})$.

We use the notation $\lesssim^{IF}$ in $IFN(\mathbb{R})$ to mean

$$\tilde{a} \lesssim^{IF} \tilde{b} \Leftrightarrow \Phi(\tilde{a}) \leqq \Phi(\tilde{b}),$$

with at least one component in Φ having strict inequality. In other words,

$$\tilde{a} \lesssim^{IF} \tilde{b} \Leftrightarrow \Phi(\tilde{b}) - \Phi(\tilde{a}) \in \mathbb{R}^{2r+2s}_{+} \setminus \{0\}.$$

If $\tilde{a} = [(175, 5, 5), (175, 8, 10)]$ and $\tilde{b} = [(176, 6, 7), (176, 7, 10)]$, then $\Phi(\tilde{a}) = (170, 180, 175, 167, 185)$ and $\Phi(\tilde{b}) = (170, 183, 176, 169, 186)$; obviously, $\Phi(\tilde{a}) \lesssim^{IF} \Phi(\tilde{b})$.

4. Formulation of Matrix Game with I-fuzzy Payoffs

Let $A \in \mathbb{R}^{m\times n}$ be an $m\times n$ real matrix. A crisp TPZSMG G is a triplet $G = (S^m, S^n, A)$, where $S^m = \{x \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} x_i = 1\}$ and $S^n = \{y \in \mathbb{R}^n_+ \mid \sum_{j=1}^{n} y_j = 1\}$ respectively denote the strategy spaces for Player I and Player II, and $A = [a_{lk}]$, $l = 1, \cdots, m$, $k = 1, \cdots, n$, is the payoff matrix. Here it is to be noted that x_i is the probability that the Player I will choose i^{th} pure strategy and y_j is the probability that the Player II will choose j^{th} pure strategy. Whenever Player I chooses the mixed strategy $x \in S^m$ and Player II chooses the mixed strategy $y \in S^n$, then expected payoff to the game is $E(x, y) = x^T Ay = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_i y_j$. Since the game is a zero sum game, $x^T Ay$ and $-x^T Ay$ are the expected pay-offs for Player I and Player II respectively.

4.1. POSS

Let $(IFMG)^{IF}_{\precsim} = (S^m, S^n, \tilde{A})$ be a class of I-fuzzy two person matrix game, with I-fuzzy order $\precsim^{IF}$, to be studied in the sequel. Here S^m and S^n are as aforementioned. The entries in the payoff matrix $\tilde{A}$ are I-fuzzy numbers. It is thus natural that the expected payoff for each player is an I-fuzzy number. For each pair of strategies $(x, y) \in S^m \times S^n$, we shall be using the notations $\widetilde{E(x, y)} = \Phi(x^T \tilde{A}y)$, where the description of $\Phi(x^T \tilde{A}y)$ is explained below, to denote the expected value of the game. Since the game is a zero-sum game, $\widetilde{E(x, y)}$ and $-\widetilde{E(x, y)}$ are expected payoffs for Player I and Player II respectively.

Observe that for $\tilde{A} = [\tilde{a}_{lk}]$, $\gamma_\mu = \{\alpha_1, \cdots, \alpha_r\}$ and $\gamma_\nu = \{\beta_1, \cdots, \beta_s\}$, the (α, β)-cuts are written as $(\tilde{A}^j_{\alpha_i})^\mu = [(a_{lk}{}^j_{\alpha_i})^\mu]$ and $(\tilde{A}^j_{\beta_t})^\nu = [(a_{lk}{}^j_{\beta_t})^\nu]$, $i = 1, \cdots, r$, $t = 1, \cdots, s$, $j = 1, 2$. By $\Phi(x^T \tilde{A}y)$ we mean a vector in $\mathbb{R}^{2r+2s}$ given by

$$[x^T (\tilde{A}^1_{\alpha_i})^\mu y \ \ (i = 1, \cdots, r),\ x^T (\tilde{A}^2_{\alpha_i})^\mu y \ \ (i = 1, \cdots, r),$$
$$x^T (\tilde{A}^1_{\beta_t})^\nu y \ \ (t = 1, \cdots, s),\ x^T (\tilde{A}^2_{\beta_t})^\nu y \ \ (t = 1, \cdots, s)].$$

Remark 4.1 The important and challenging problem is how to select the cut points α_i and β_t in the above concept? Though at present, we do not have any scheme for making theses choices, however intuitively the cut points can be chosen so as they properly describe the shape of the membership functions (for α_i) and the non-membership functions (for β_t) of the I-fuzzy numbers entries in $\tilde{A}$. For instance, if we encounter the triangular or trapezoidal I-fuzzy numbers, then the obvious choice for the α_i-cuts are 0-cut and 1-cut and similar choice for the β_t-cuts. If the I-fuzzy numbers have piecewise linear shapes, then the break-points in them can help us to appropriately select α_i and β_t, something that we have also demonstrated in an example included in Section 5. On the other hand, we can see the slope or shape changing points in the membership and the non-membership functions of the I-fuzzy numbers to learn the cut points. But indeed, this is a question to address as to how many and precisely what values of the cut-points will be appropriate to generate a reasonably good approximation of the value of the game $(IFMG)^{IF}_{\precsim}$.

Let us return back to the game $(IFMG)^{IF}_{\precsim}$ and define the solution concept for it. A strategy profile (x^*, y^*) is a Pareto-Nash equilibrium strategy or simply Pareto equilibrium strategy [27] of two players if deviations from the equilibrium strategies do not offer any gains, in the sense of Pareto, to any of the payoff functions for any of the players.

Definition 4.1 ((γ_μ, γ_ν)-Pareto-Nash equilibrium strategy) *A pair* $(x^*, y^*) \in S^m \times S^n$ *is a Nash equilibrium strategy for a game* $(IFMG)^{IF}_{\precsim}$ *if there does not exists any* $(x, y) \in S^m \times S^n$ *such that the following hold:*

(i) $\Phi(x^T\, \tilde{A}\, y^*) - \Phi((x^*)^T\, \tilde{A}\, y^*) \quad \in \mathbb{R}_+^{2r+2s} \setminus \{0\}$,

(ii) $\Phi((x^*)^T\, \tilde{A}\, y^*) - \Phi((x^*)^T\, \tilde{A}\, y) \quad \in \mathbb{R}_+^{2r+2s} \setminus \{0\}$.

We now define security levels and POSS for each player.

Remark 4.2 If the payoff matrix $\tilde{A} = A \in \mathbb{R}^{m\times n}$ (that is a crisp matrix), then Definition 4.1 of Pareto-Nash equilibrium strategy subsume to the Nash equilibrium strategies for a crisp TPZSMG.

Definition 4.2 (Security level for Player I) *For a strategy* $x \in S^m$, *the security level of Player* I *is an I-fuzzy number* $\widetilde{v(x)} \in IFN(\mathbb{R})$, *such that the application of the aforementioned ranking function* Φ *to it yields*

$$\begin{aligned} g(\widetilde{v(x)}) &= \left(\inf_{y\in S^n}(p^\mu_{ij}(\widetilde{E(x,y)}))\ \ (i = 1,\cdots,r,\ \ j = 1,2)\right) \\ &= \left(\inf_{y\in S^n} x^T(\tilde{A}^j_{\alpha_i})^\mu y\ \ (i = 1,\cdots,r,\ \ j = 1,2)\right) \in \mathbb{R}^{2r}, \\ h(\widetilde{v(x)}) &= \left(\inf_{y\in S^n}(q^\nu_{tj}(\widetilde{E(x,y)}))\ \ (t = 1,\cdots,s,\ \ j = 1,2)\right) \\ &= \left(\inf_{y\in S^n} x^T(\tilde{A}^j_{\beta_t})^\nu y\ \ (t = 1,\cdots,s,\ \ j = 1,2)\right) \in \mathbb{R}^{2s}. \end{aligned}$$

Definition 4.3 (Security level for Player II) *For a strategy* $y \in S^n$, *the security level of Player* II *is an I-fuzzy number* $\widetilde{w(y)} \in IFN(\mathbb{R})$ *such that the application of the aforementioned ranking function* Φ *to it yields*

$$\begin{aligned} g(\widetilde{w(y)}) &= \left(\sup_{x\in S^m}(p^\mu_{ij}(\widetilde{E(x,y)}))\ \ (i = 1,\cdots,r,\ \ j = 1,2)\right) \\ &= \left(\sup_{x\in S^m} x^T(\tilde{A}^j_{\alpha_i})^\mu y\ \ (i = 1,\cdots,r,\ \ j = 1,2)\right) \in \mathbb{R}^{2r}, \\ h(\widetilde{w(y)}) &= \left(\sup_{x\in S^m}(q^\nu_{tj}(\widetilde{E(x,y)}))\ \ (t = 1,\cdots,s,\ \ j = 1,2)\right) \\ &= \left(\sup_{x\in S^m} x^T(\tilde{A}^j_{\beta_t})^\nu y\ \ (t = 1,\cdots,s,\ \ j = 1,2)\right) \in \mathbb{R}^{2s}. \end{aligned}$$

Note that the infimum and the supremum are taken componentwise in all expressions above.

The functions $g(\widetilde{v(x)})$ and $g(\widetilde{w(y)})$ enable us to construct the membership functions of $\widetilde{v(x)}$ and $\widetilde{w(y)}$ respectively, while $h(\widetilde{v(x)})$ and $h(\widetilde{w(y)})$ provide the non-membership functions of $\widetilde{v(x)}$ and $\widetilde{w(y)}$ respectively.

Proposition 4.1 *The security levels $\widetilde{v(x)}$ and $\widetilde{w(y)}$ for Player* I *and Player* II *respectively are well defined I-fuzzy numbers.*

Proof As all the entries of the payoff matrix are I-fuzzy numbers, therefore we have

$$(\tilde{A}^1_{\alpha_i})^\mu \leqq (\tilde{A}^2_{\alpha_i})^\mu, \ \ (\tilde{A}^1_{\beta_t})^\nu \leqq (\tilde{A}^2_{\beta_t})^\nu \, , \ i = 1, \cdots, r, \ t = 1, \cdots, s.$$

Therefore, for any $x \in S^m$ and $y \in S^n$,

$$x^T (\tilde{A}^1_{\alpha_i})^\mu y \leqq x^T (\tilde{A}^2_{\alpha_i})^\mu y \, , \ \ i = 1, \cdots, r,$$

and

$$x^T (\tilde{A}^1_{\beta_t})^\nu y \leqq x^T (\tilde{A}^2_{\beta_t})^\nu y \, , \ t = 1, \cdots, s,$$

yielding

$$(\widetilde{v(x)}^1_{\alpha_i})^\mu \leqq (\widetilde{v(x)}^2_{\alpha_i})^\mu, \ \ (\widetilde{v(x)}^1_{\beta_t})^\nu \leqq (\widetilde{v(x)}^2_{\beta_t})^\nu \, , \ i = 1, \cdots, r, \ t = 1, \cdots, s.$$

Now if $\alpha_i < \alpha_i^{'}$ and $\beta_t < \beta_t^{'}$, then

$$(\tilde{A}^1_{\alpha_i})^\mu \leqq (\tilde{A}^1_{\alpha_i^{'}})^\mu, \ \ (\tilde{A}^2_{\alpha_i})^\mu \geqq (\tilde{A}^2_{\alpha_i^{'}})^\mu \, ,$$

and

$$(\tilde{A}^1_{\beta_t})^\nu \leqq (\tilde{A}^1_{\beta_t^{'}})^\nu, \ \ (\tilde{A}^2_{\beta_t})^\nu \geqq (\tilde{A}^2_{\beta_t^{'}})^\nu.$$

Therefore, for $x \in S^m$, we get

$$(\widetilde{v(x)}^1_{\alpha_i})^\mu \leqq \widetilde{(}v(x)^1_{\alpha_i^{'}})^\mu, \ \ (\widetilde{v(x)}^2_{\alpha_i})^\mu \geqq (\widetilde{v}^2_{\alpha_i^{'}})^\mu \, ,$$

and

$$(\widetilde{v(x)}^1_{\beta_t})^\nu \leqq (\widetilde{v(x)}^1_{\beta_t^{'}})^\nu, \ \ (\widetilde{v(x)}^2_{\beta_t})^\nu \geqq (\widetilde{v(x)}^2_{\beta_t^{'}})^\nu.$$

Similar inequalities can easily be worked out for $\widetilde{w(y)}$. Thus, for any $x \in S^m$ and $y \in S^n$, the α-cuts of $\widetilde{v(x)}$ and the β-cuts of $\widetilde{w(y)}$ are nested. The requisite result hence follows.

Next we propose to analyze $(IFMG)^{IF}_{\precsim}$ in the worst case behavior of the opponents. It thus makes sense to choose that strategy for Player I (and II) that maximizes (and minimizes) his security level. Here, maximization (or minimization) is to be understood in terms of finding maximal (or minimal) elements with respect to the partial ordering induced by the I-fuzzy order $\precsim^{IF}$ (or $\succsim^{IF}$), leading to the following definitions.

Definition 4.4 (POSS for Player I) *A strategy $x^* \in S^m$ is a POSS for Player* I *if there is no $x \in S^m$ such that*

$$\widetilde{v(x^*)} \precsim^{IF} \widetilde{v(x)}.$$

Definition 4.5 (POSS for Player II) *A strategy $y^* \in S^n$ is a POSS for Player* II *if there is no $y \in S^n$ such that*

$$\widetilde{w(y^*)} \succsim^{IF} \widetilde{w(y)}.$$

Recall, $\tilde{a} \precsim^{IF} \tilde{b}$ means $\Phi(\tilde{a}) \leqq \Phi(\tilde{b})$ (componentwise) with at least one component having strictly less inequality, and $\tilde{a} \succsim^{IF} \tilde{b}$ is taken as $-\tilde{a} \precsim^{IF} -\tilde{b}$.

If x^* is a POSS for Player I, then his security level is given by $\widetilde{v^*} = \widetilde{v(x^*)}$. Similarly if y^* is a POSS for Player II, then his security level is given by $\widetilde{w^*} = \widetilde{w(y^*)}$.

4.2. Computing POSS for Players

We now discuss a procedure to compute POSS for both players. In this regard, we have the following theorems on the lines of Clemente et al. [12].

Theorem 4.1 *The strategy x^* is a POSS and $\widetilde{v^*}$ is the security level for Player* I *if and only if $(x^*, \widetilde{v^*})$ is an efficient solution to the following multiobjective programming problem:*

$$\begin{array}{lll}
(VP)_1 & \max & \Big((v^j_{\alpha_i})^\mu,\ (v^j_{\beta_t})^\nu\ \ (i = 1, \cdots, r,\ t = 1, \cdots, s,\ j = 1, 2)\Big) \\
& \text{subject to} & \\
& & x^T(\tilde{A}^j_{\alpha_i})^\mu \ \geqq\ (v^j_{\alpha_i})^\mu e_n, \quad i = 1, \cdots, r,\ j = 1, 2, \\
& & x^T(\tilde{A}^j_{\beta_t})^\nu \ \geqq\ (v^j_{\beta_t})^\nu e_n, \quad t = 1, \cdots, s,\ j = 1, 2, \\
& & x \in S^m,
\end{array}$$

where $e_n = (1, \cdots, 1)^T \in \mathbb{R}^n$.

Proof Let x^* be a POSS for $(IFMG)^{IF}_{\precsim}$. This means that there is no $x \in S^m$ such that

$$\widetilde{v(x^*)} \precsim^{IF} \widetilde{v(x)},$$

that is, there is no $x \in S^m$ such that $\Phi(\widetilde{v(x^*)}) \leqq \Phi(\widetilde{v(x)})$ with at least one component having strict inequality.

From the definition of security level for Player I

$$\begin{aligned} g(\widetilde{v(x)}) &= (\inf_{y\in S^n}(x^T(\tilde{A}^j_{\alpha_i})^{\mu}\, y)\ \ (i=1,\cdots,r,\ j=1,2)) \\ &= (\min_{u=1,\cdots,n}(x^T(\tilde{A}^j_{\alpha_i})^{\mu}\, u)\ \ (i=1,\cdots,r,\ j=1,2)), \end{aligned}$$

and

$$\begin{aligned} h(\widetilde{v(x)}) &= (\inf_{x\in S^m}(x^T(\tilde{A}^j_{\beta_t})^{\nu}\, y)\ \ (t=1,\cdots,s,\ j=1,2)) \\ &= (\min_{u=1,\cdots,n}(x^T(\tilde{A}^j_{\beta_t})^{\nu}\, u)\ \ (t=1,\cdots,s,\ j=1,2)), \end{aligned}$$

where $u=(0,\cdots,1_u,\cdots,0)^T$ and recall $\Phi(\,\widetilde{v(x)}\,)=(\,g(\widetilde{v(x)}),\ h(\widetilde{v(x)})\,)$.

Therefore, there does not exist any $x\in S^m$ such that

$$(\,g(\widetilde{v(x^*)}),\ h(\widetilde{v(x^*)})\,)\leqq(\,g(\widetilde{v(x)}),\ h(\widetilde{v(x)})\,)$$

with at least one component having strict inequality.

This implies that x^* is an efficient solution to the problem

$$\max_{x\in S^n}\Big(\big(\min_{u=1,\cdots,n} x^T(\tilde{A}^j_{\alpha_i})^{\mu}u,\ \min_{u=1,\cdots,n} x^T(\tilde{A}^j_{\beta_t})^{\nu}u\big)\,(i=1,\cdots,r,\ t=1,\cdots,s,\ j=1,2)\Big)$$

which is equivalent to $(VP)_1$ with

$$(v^j_{\alpha_i})^{\mu}=\min_{u=1,\cdots,n} x^T(\tilde{A}^j_{\alpha_i})^{\mu}u\ \ \text{and}\ \ (v^j_{\beta_t})^{\nu}=\min_{u=1,\cdots,n} x^T(\tilde{A}^j_{\beta_t})^{\nu}u.$$

The requisite result follows.

Theorem 4.2 *The strategy y^* is a POSS and $\widetilde{w^*}$ is the security level for Player* II *if and only if $(y^*,\widetilde{w^*})$ is the efficient solution to the following multiobjective programming problem:*

$$\begin{aligned} (VP)_2\qquad & \min\ \Big((w^j_{\alpha_i})^{\mu},\ (w^j_{\beta_t})^{\nu}\ \ (i=1,\cdots,r,\ t=1,\cdots,s,\ j=1,2)\Big) \\ & \text{subject to} \\ & \qquad (\tilde{A}^j_{\alpha_i})^{\mu}y \leqq (w^j_{\alpha_i})^{\mu}e_m,\quad i=1,\cdots r,\ j=1,2, \\ & \qquad (\tilde{A}^j_{\beta_t})^{\nu}y \leqq (w^j_{\beta_t})^{\nu}e_m,\quad t=1,\cdots,s,\ j=1,2, \\ & \qquad y\in S^n. \end{aligned}$$

The proof can easily be worked out on similar lines as the proof of previous theorem.

Theorem 4.3 *If $(x^*,\widetilde{v^*})$ is an efficient solution to $(VP)_1$ and $(y^*,\widetilde{w^*})$ is an efficient solution to $(VP)_2$, then the security levels satisfy the following weak duality relation*

$$\widetilde{v^*}\precsim^{IF}\widetilde{w^*}.$$

Proof follows from the feasibility of $(x^*,\widetilde{v^*})$ for $(VP)_1$ and $(y^*,\widetilde{w^*})$ for $(VP)_2$.

We note that in the above theorem it is enough to assume feasibility of respective solutions to $(VP)_1$ and $(VP)_2$.

Theorem 4.4 (i) *If* $(x^*, \widetilde{v^*})$ *and* $(y^*, \widetilde{w^*})$ *are efficient solutions to* $(VP)_1$ *and* $(VP)_2$*, respectively, then* (x^*, y^*) *is* (γ_μ, γ_ν)*-Pareto-Nash equilibrium strategy for the game* $(IFMG)^{IF}_{\precsim}$.

(ii) (x^*, y^*) *is* (γ_μ, γ_ν)*-Pareto-Nash equilibrium strategy for the game* $(IFMG)^{IF}_{\precsim}$*, then* x^* *and* y^* *are efficient solutions to* $(VP)_1$ *and* $(VP)_2$*, respectively.*

The proof is omitted here as it obviously follows from the Definitions of Pareto-Nash equilibrium strategy and POSS.

5. Methodology and Its Illustration

All the multiobjective programming problems, resulting in the game models of the presented examples in Section 5.2 and Appendix, are solved by the Augmented ϵ-constraint (AUGMECON) method [20, 21] implemented on GAMS [14]. We have used the MOSEK solver within 64 bits Window 7 version of GAMS.

5.1. Implementation Issues

The AUGMECON method, first find the range of each objective function over the constraint set (or feasible set). The range of each of the j^{th}-objective function, $j = 1, \cdots, \omega$, where ω is the total number of objective functions, is divided into q_j equal intervals using $(g_j + 1)$ intermediate equidistant grid points. Thereafter, the method constructs an optimization problem where, besides the original constraints, all objective functions except first are also taken into constraints with their right hand side resource parameters constructed using the generated grid points. The total number of runs becomes $\prod_{j=2}^{\omega}(g_j + 1)$. The detailed description of the efficient frontier can be obtained if the method above have large numbers of grid points, but then it also means higher computation time; so a trade-off can be worked out between the number of points one wishes to generate on the efficient frontier and the time of their computation. Interested readers can find detail information, working and analysis of the AUGMECON method in [21].

Although the aforementioned method is robust to handle several objective functions simultaneously and generate efficient frontier with high density, yet it is a desirable characteristic of the method to control the number of grid points (else the computation time can be enormous) and these grid points are generated using $\omega - 1$ objective functions; consequently leading to a trade-off between the number of objective functions and the density of the efficient frontier of the problem. Thus, if the multiobjective program has several objective functions then computing many efficient solutions for it can be a costly affair.

In our study, the number of objective functions in the multiobjective programs $(VP)_1$ and $(VP)_2$ depend on the α_i-cuts and β_t-cuts. Thus, the more these cut points are the better will be the representation of security levels for both players. But then this also means several objective functions in $(VP)_1$ and $(VP)_2$ leading to a trade-off between number of efficient points (viz. the POSS for players) of respective problems and their computation time. The practitioners are thus required to tune in between the cuts, reasonable representation of the security levels, generation of number of efficient points, and computational time.

5.2. Numerical Example

The following example is a simple variations of the original 2×2 matrix game constructed by Campos [11]. It may be pointed out that Campos' example more or less has become a benchmark example to illustrate various approaches for solving fuzzy matrix games. Now the entries of the payoff matrix are I-fuzzy numbers, the membership and the nonmembership functions have been prescribed separately corresponding to each entry in the payoff matrix.

Example 5.1 Consider the I-fuzzy matrix game with payoffs matrix as follows:

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix},$$

where the entries $\tilde{a}_{ij}$ as TIFNs given by

$$\tilde{a}_{11} = [[180, 5, 10], [180, 10, 15]], \quad \tilde{a}_{12} = [[156, 6, 2], [156, 11, 7]],$$
$$\tilde{a}_{21} = [[90, 10, 10], [90, 15, 15]], \quad \tilde{a}_{22} = [[180, 5, 10], [180, 10, 15]].$$

In TIFN, $\tilde{a}_{ij}$, the first triplet represents membership $\mu_{\tilde{a}_{ij}}$ and the second triplet represents non-membership $\nu_{\tilde{a}_{ij}}$.

We take $\{\alpha_1 = 0,\ \alpha_2 = 1\}$ and $\{\beta_1 = 1,\ \beta_2 = 0\}$. This yields

$$(\tilde{A}_0^1)^\mu = \begin{pmatrix} 175 & 150 \\ 80 & 175 \end{pmatrix}, \quad (\tilde{A}_0^2)^\mu = \begin{pmatrix} 190 & 158 \\ 100 & 190 \end{pmatrix},$$

$$(\tilde{A}_1^1)^\mu = (\tilde{A}_1^2)^\mu = (\tilde{A}_1^1)^\nu = (\tilde{A}_1^2)^\nu = \begin{pmatrix} 180 & 156 \\ 90 & 180 \end{pmatrix},$$

$$(\tilde{A}_0^1)^\nu = \begin{pmatrix} 170 & 145 \\ 75 & 170 \end{pmatrix}, \quad (\tilde{A}_0^2)^\nu = \begin{pmatrix} 195 & 163 \\ 105 & 195 \end{pmatrix}.$$

We need to solve the following multiobjective linear programming problem for Player I.

$$\begin{array}{ll} \max & \left((v_0^1)^\mu, (v_0^2)^\mu, (v_1^1)^\mu, (v_0^1)^\nu, (v_0^2)^\nu\right) \\ \text{subject to} & \\ & \begin{array}{ll} 175x_1 + 80x_2 \geqq (v_0^1)^\mu, & 150x_1 + 175x_2 \geqq (v_0^1)^\mu, \\ 190x_1 + 100x_2 \geqq (v_0^2)^\mu, & 158x_1 + 190x_2 \geqq (v_0^2)^\mu, \\ 180x_1 + 90x_2 \geqq (v_1^1)^\mu, & 156x_1 + 180x_2 \geqq (v_1^1)^\mu, \\ 170x_1 + 75x_2 \geqq (v_0^1)^\nu, & 145x_1 + 170x_2 \geqq (v_0^1)^\nu, \\ 195x_1 + 105x_2 \geqq (v_0^2)^\nu, & 163x_1 + 195x_2 \geqq (v_0^2)^\nu, \\ x_1 + x_2 = 1, & x_1, x_2 \geqq 0. \end{array} \end{array}$$

One of the efficient solution (viz. the POSS) is $x_1^* = 0.74,\ x_2^* = 0.26$ with

$$\Big((v_0^{*1})^\mu = 150.080,\ (v_0^{*2})^\mu = 166.390,\ (v_1^{*1})^\mu = 156.390\Big)$$

and

$$\Big((v_0^{*1})^\nu = 145.080,\ (v_0^{*2})^\nu = 171.390\Big).$$

Therefore $(x_1^* = 0.74,\ x_2^* = 0.26)$ is a POSS for Player I and his value is the I-fuzzy number

$$[[156.390, 6.31, 10.00], [156.390, 11.31, 15]]\,.$$

This strategy is some what different from that of Clemente et al. [12] simply because we are working with I-fuzzy payoffs while Clemente et al. [12] worked with fuzzy payoffs. If ν is taken as $1-\mu$, then our results coincide with that of Clemente et al. [12].

In fact, we can generate the set of efficient solutions (viz. POSS) for the above multiobjective program. Below we list some of them and the corresponding security levels for Player I.

Table 1: POSS and security levels for Player I.

x_1^*	x_2^*	$\widetilde{v(x*)}$
0.74	0.26	[(156.39,6.31,10),(156.39,11.31,15)]
0.76	0.24	[(158.39,5.74,7.14),(158.39,10.74,12.14)]
0.79	0.21	[(161,5.79,4.67),(161,10.79,8.67)]

In a similar manner, we need to solve the following multiobjective linear programming problem for obtaining the POSS for Player II.

$$\begin{aligned}
&\min \quad \Big((w_0^1)^\mu, (w_0^2)^\mu, (w_1^1)^\mu, (w_0^1)^\nu, (w_0^2)^\nu\Big)\\
&\text{subject to}\\
&\qquad 175y_1 + 150y_2 \leqq (w_0^1)^\mu\,, \qquad 80y_1 + 175y_2 \leqq (w_0^1)^\mu\,,\\
&\qquad 190y_1 + 158y_2 \leqq (w_0^2)^\mu\,, \qquad 100y_1 + 190y_2 \leqq (w_0^2)^\mu\,,\\
&\qquad 180y_1 + 156y_2 \leqq (w_1^1)^\mu\,, \qquad 90y_1 + 180y_2 \leqq (w_1^1)^\mu\,,\\
&\qquad 170y_1 + 145y_2 \leqq (w_0^1)^\nu\,, \qquad 75y_1 + 170y_2 \leqq (w_0^1)^\nu\,,\\
&\qquad 195y_1 + 163y_2 \leqq (w_0^2)^\nu\,, \qquad 105y_1 + 195y_2 \leqq (w_0^2)^\nu\,,\\
&\qquad y_1 + y_2 = 1\,, \qquad y_1, y_2 \geqq 0.
\end{aligned}$$

One of its efficient solutions (viz. the POSS) is $y_1^* = 0.21,\ y_2^* = 0.79$ with

$$\Big((w_0^{*1})^\mu = 155.210, (w_0^{*2})^\mu = 171.250, (w_1^{*1})^\mu = 161.250\Big)$$

and

$$\Big((w_1^{*1})^\nu = 150.210, (w_1^{*2})^\nu = 176.250, (w_1^{*1})^\nu = 161.250\Big).$$

Therefore $(y_1^* = 0.21,\ y_2^* = 0.79)$ is a POSS for Player II and his value is the I-fuzzy number

$$[[161.25, 6.04, 10.0], [161.65, 11.04, 15.00]].$$

Below we list some more POSS for Player II.

Table 2: POSS and security levels for Player II.

y_1^*	y_2^*	$\widetilde{w(y*)}$
0.21	0.79	[(161.25,6.04,10),(161.25,11.04,15)]
0.24	0.76	[(161.65,5.77,7.17),(161.65,10.77,12.17)]
0.26	0.74	[(162.3,6.74,4.09),(162.3,10.74,9.09)]

Remark 5.1 It is worth pointing out that the average indexes of the optimal I-fuzzy values of the two players listed above turn out to be the same as when we solve the problems of players by Nan and Li [23] approach of optimizing average indexes. In fact, once we have a complete description of membership functions and non-membership functions of optimal I-fuzzy values of the game, we can easily compute the optimal average indexes; but reverse is obviously not possible. This clearly demonstrate the advantage of our approach over Nan and Li [23] method.

Remark 5.2 If I-fuzzy numbers $\tilde{a}_{ij}$ have piecewise linear membership functions and non-membership functions, then we can easily choose α_i, $i = 1, \cdots, r$, and β_t, $t = 1, \cdots, s$, to get complete description of these functions in terms of their corresponding (α, β)-cuts. An obvious choice of such cut points is the break points in the piecewise linear graphs of these functions. An example to this effect is presented in Appendix where the payoffs are piecewise linear I-fuzzy numbers. It is important to note that such a game can not be solved by Nan and Li [23] approach.

Remark 5.3 If I-fuzzy numbers $\tilde{a}_{ij}$ have in general bell shaped nonlinear membership and non-membership functions, then again we can suitably choose (and one such choice is mean $\pm\rho$ standard deviation, $\rho = 1, 2, 3$) α_i, $i = 1, \cdots, r$, and β_t, $t = 1, \cdots, s$, to get an approximate description of these functions in terms of their corresponding (α, β)-cuts. Thus we can use the same methodology to get an approximate solution for an I-fuzzy matrix game. This approximation can be improved by taking larger values for r and s.

Remark 5.4 A Pareto-optimal solutions of average indexes corresponding to membership function and non-membership function of I-fuzzy values of game were obtained in [23]. In present work, we have generated an efficient frontier of I-fuzzy values of the game, so a much more information about optimal solutions and optimal values is revealed.

6. Conclusion

A recent study of Clemente et al. [12] is generalized to the I-fuzzy matrix games having payoffs described by I-fuzzy numbers. We first extend the standard ranking order of González and Vila [15, 16] to I-fuzzy numbers and then introduce the concept of POSS for such I-fuzzy matrix games. The computation of POSS for each player has become very handy because it only requires solving an equivalent multiobjective programming problem for that player. Our approach, as presented here, provides an efficient solution to I-fuzzy matrix games with piecewise linear membership and nonmembership functions.

Though this study is very much in the sprit of Clemente et al. [12], we have a certain added advantage on the solutions over that of Clemente et al. [12] in the form of membership and non-membership functions for the payoffs of both players.

Mijanur et al. [22] extended the work of Li [18] to solve two person bi-matrix game with payoffs TIFNs. An interesting future study in this direction could be to study POSS for bi-matrix games and *n*-person games.

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Appendix

An example is presented for a more comprehensive understanding of the approach of (α,β)-cuts in a game with I-fuzzy numbers having piecewise linear membership and nonmembership functions.

Consider the I-fuzzy matrix game with payoff matrix as

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{pmatrix}.$$

The entries $\tilde{a}_{ij}$ are piecewise linear trapezoidal I-fuzzy numbers, given as follows:

$$\mu_{\tilde{a}_{11}}(x) = \begin{cases} 0\,, & x < 175\,, \\ 0.6(\dfrac{x-175}{2})\,, & 175 < x \leqq 177\,, \\ 0.6 + 0.4(\dfrac{x-177}{3})\,, & 177 < x \leqq 180\,, \\ 1\,, & 180 < x \leqq 185\,, \\ 0.6 + 0.4(\dfrac{188-x}{3})\,, & 185 < x \leqq 188\,, \\ 0.6(\dfrac{190-x}{2})\,, & 188 < x \leqq 190\,, \\ 0\,, & x \geqq 190\,, \end{cases}$$

$$\nu_{\tilde{a}_{11}}(x) = \begin{cases} 1\,, & x \leqq 170\,, \\ \dfrac{180-x}{10}\,, & 170 < x \leqq 180\,, \\ 0\,, & 180 < x \leqq 185\,, \\ \dfrac{x-185}{10}\,, & 185 < x \leqq 195\,, \\ 1\,, & x \geqq 195\,, \end{cases}$$

$$\mu_{\tilde{a}_{12}}(x) = \begin{cases} 0\,, & x < 150\,, \\ \dfrac{x-150}{6}\,, & 150 < x \leqq 156\,, \\ 1\,, & 156 < x \leqq 157\,, \\ \dfrac{157-x}{1}\,, & 157 < x \leqq 158\,, \\ 0\,, & x \geqq 158\,, \end{cases}$$

$$\nu_{\tilde{a}_{12}}(x) = \begin{cases} 1\,, & x \leqq 145\,, \\ \dfrac{156 - x}{11}\,, & 145 < x \leqq 156\,, \\ 0\,, & 156 < x \leqq 157\,, \\ \dfrac{x - 157}{6}\,, & 157 < x \leqq 163\,, \\ 1\,, & x \geqq 163\,, \end{cases}$$

$$\mu_{\tilde{a}_{21}}(x) = \begin{cases} 0\,, & x < 80\,, \\ 0.7(\dfrac{x - 80}{5})\,, & 80 < x \leqq 85\,, \\ 0.7 + 0.3(\dfrac{x - 85}{5})\,, & 85 < x \leqq 90\,, \\ 1\,, & 90 < x \leqq 94\,, \\ 0.7 + 0.3(\dfrac{97 - x}{3})\,, & 94 < x \leqq 97\,, \\ 0.7(\dfrac{100 - x}{3})\,, & 97 < x \leqq 100\,, \\ 0\,, & x \geqq 100\,, \end{cases}$$

$$\nu_{\tilde{a}_{21}}(x) = \begin{cases} 1\,, & x \leqq 65\,, \\ \dfrac{90 - x}{25}\,, & 65 < x \leqq 90\,, \\ 0\,, & 90 < x \leqq 94\,, \\ \dfrac{x - 94}{15}\,, & 94 < x \leqq 109\,, \\ 1\,, & x \geqq 109\,, \end{cases}$$

$$\mu_{\tilde{a}_{22}}(x) = \begin{cases} 0\,, & x < 175\,, \\ \dfrac{x - 175}{5}\,, & 175 < x \leqq 180\,, \\ 1\,, & 180 < x \leqq 185\,, \\ \dfrac{190 - x}{5} & 185\,, < x \leqq 190\,, \\ 0\,, & x \geqq 190\,, \end{cases}$$

$$\nu_{\widetilde{a}_{22}}(x)=\begin{cases}1\,, & x\leqq 170\,,\\ \dfrac{180-x}{10}\,, & 170<x\leqq 180\,,\\ 0\,, & 180<x\leqq 185\,,\\ \dfrac{x-185}{10}\,, & 185<x\leqq 195\,,\\ 1\,, & x\geqq 195.\end{cases}$$

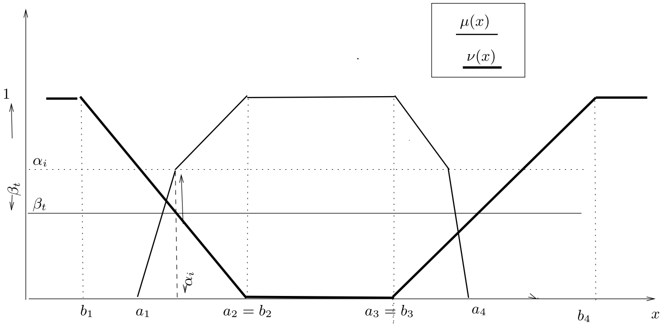

Fig. 2 Membership and non-membership functions.

As the information is prescribed in terms of piecewise trapezoidal I-fuzzy numbers we again take $\{\alpha_1=0,\ \alpha_2=0.6,\ \alpha_3=0.7,\ \alpha_4=1\}$ and $\{\beta_1=1,\ \beta_2=0.8,\ \beta_3=0.7,\ \beta_4=0\}$. A general depiction of such an (α,β)-cut is depicted in Figure . We thus have the following:

$$(\tilde{A}^1_0)^\mu=\begin{pmatrix}175 & 150\\ 80 & 175\end{pmatrix},\qquad (\tilde{A}^2_0)^\mu=\begin{pmatrix}190 & 158\\ 100 & 190\end{pmatrix},$$

$$(\tilde{A}^1_{0.6})^\mu=\begin{pmatrix}177 & 153.6\\ 84.29 & 178\end{pmatrix},\qquad (\tilde{A}^2_{0.6})^\mu=\begin{pmatrix}188 & 156.4\\ 97.43 & 187\end{pmatrix},$$

$$(\tilde{A}^1_{0.7})^\mu=\begin{pmatrix}177.75 & 154.2\\ 85 & 178.5\end{pmatrix},\qquad (\tilde{A}^2_{0.6})^\mu=\begin{pmatrix}187.25 & 156.3\\ 97 & 186.5\end{pmatrix},$$

$$(\tilde{A}^1_1)^\mu=(\tilde{A}^1_1)^\nu=\begin{pmatrix}180 & 156\\ 90 & 180\end{pmatrix},\qquad (\tilde{A}^2_1)^\mu=(\tilde{A}^2_1)^\nu=\begin{pmatrix}185 & 157\\ 94 & 185\end{pmatrix},$$

$$(\tilde{A}^1_{0.8})^\nu = \begin{pmatrix} 178 & 153.8 \\ 85 & 178 \end{pmatrix}, \qquad (\tilde{A}^2_{0.8})^\nu = \begin{pmatrix} 187 & 158.2 \\ 97 & 187 \end{pmatrix},$$

$$(\tilde{A}^1_{0.7})^\nu = \begin{pmatrix} 177 & 152.7 \\ 82.5 & 177 \end{pmatrix}, \qquad (\tilde{A}^2_{0.7})^\nu = \begin{pmatrix} 188 & 158.8 \\ 98.5 & 188 \end{pmatrix},$$

$$(\tilde{A}^1_{0})^\nu = \begin{pmatrix} 170 & 145 \\ 65 & 170 \end{pmatrix}, \qquad (\tilde{A}^2_{0})^\nu = \begin{pmatrix} 195 & 163 \\ 109 & 195 \end{pmatrix}.$$

To solve the game for Player I, we need to solve the following multiobjective linear programming problem:

$$\begin{aligned}
\max \quad & \Big((v^1_0)^\mu,\ (v^2_0)^\mu,\ (v^1_{0.6})^\mu,\ (v^2_{0.6})^\mu,\ (v^1_{0.7})^\mu,\ (v^2_{0.7})^\mu,\ (v^1_1)^\mu,\ (v^2_1)^\mu, \\
& (v^1_{0.8})^\nu,\ (v^2_{0.8})^\nu,\ (v^1_{0.7})^\nu,\ (v^2_{0.7})^\nu,\ (v^1_0)^\nu,\ (v^2_0)^\nu\Big)
\end{aligned}$$

subject to

$$\begin{array}{ll}
175x_1 + 80x_2 \geqq (v^1_0)^\mu\,, & 150x_1 + 175x_2 \geqq (v^1_0)^\mu\,, \\
190x_1 + 100x_2 \geqq (v^2_0)^\mu\,, & 158x_1 + 190x_2 \geqq (v^2_0)^\mu\,, \\
177x_1 + 84.3x_2 \geqq (v^1_{0.6})^\mu\,, & 153.6x_1 + 178x_2 \geqq (v^1_{0.6})^\mu\,, \\
188x_1 + 97.43x_2 \geqq (v^2_{0.6})^\mu\,, & 156.4x_1 + 187x_2 \geqq (v^2_{0.6})^\mu\,, \\
177.75x_1 + 85x_2 \geqq (v^1_{0.7})^\mu\,, & 154.2x_1 + 178.5x_2 \geqq (v^1_{0.7})^\mu\,, \\
187.25x_1 + 97x_2 \geqq (v^2_{0.7})^\mu\,, & 156.3x_1 + 186.5x_2 \geqq (v^2_{0.7})^\mu\,, \\
180x_1 + 90x_2 \geqq (v^1_1)^\mu\,, & 156x_1 + 180x_2 \geqq (v^1_1)^\mu\,, \\
185x_1 + 94x_2 \geqq (v^2_1)^\mu\,, & 157x_1 + 185x_2 \geqq (v^2_1)^\mu\,, \\
178x_1 + 85x_2 \geqq (v^1_{0.8})^\nu\,, & 153.8x_1 + 178x_2 \geqq (v^1_{0.8})^\nu\,, \\
187x_1 + 97x_2 \geqq (v^2_{0.8})^\nu\,, & 158.2x_1 + 187x_2 \geqq (v^2_{0.8})^\nu\,, \\
177x_1 + 82.5x_2 \geqq (v^1_{0.7})^\nu\,, & 152.7x_1 + 177x_2 \geqq (v^1_{0.7})^\nu\,, \\
188x_1 + 98.5x_2 \geqq (v^2_{0.7})^\nu\,, & 158.8x_1 + 188x_2 \geqq (v^2_{0.7})^\nu\,, \\
170x_1 + 65x_2 \geqq (v^1_0)^\nu\,, & 145x_1 + 170x_2 \geqq (v^1_0)^\nu\,, \\
195x_1 + 109x_2 \geqq (v^2_0)^\nu\,, & 163x_1 + 195x_2 \geqq (v^2_0)^\nu\,, \\
x_1 + x_2 = 1\,, & x_1, x_2 \geqq 0.
\end{array}$$

Some of the efficient solutions (viz. the POSS) with corresponding efficient values (viz. the security levels) for the above problem are listed in the following tables.

Table 3: POSS for Player I.

#	x_1^*	x_2^*
1	0.73	0.27
2	0.74	0.26
3	0.75	0.25
4	0.76	0.24

Table 4: Membership values of security levels for Player I.

#	$(v_0^{*1})^\mu$	$(v_0^{*2})^\mu$	$(v_{0.6}^{*1})^\mu$	$(v_{0.6}^{*2})^\mu$	$(v_{0.7}^{*1})^\mu$	$(v_{0.7}^{*2})^\mu$	$(v_1^{*1})^\mu$	$(v_1^{*2})^\mu$
1	149.24	165.59	151.86	163.44	152.07	162.78	155.59	160.32
2	150.08	166.39	152.69	164.24	152.91	163.58	156.39	161.13
3	150.22	166.35	152.82	164.38	153.05	163.71	156.53	161.27
4	150.59	166.22	153.18	164.26	153.42	164.06	156.87	161.62

Table 5: Non-membership values of security levels for Player I.

#	$(v_0^{*1})^\nu$	$(v_0^{*2})^\nu$	$(v_{0.7}^{*1})^\nu$	$(v_{0.7}^{*2})^\nu$	$(v_{0.8}^{*1})^\nu$	$(v_{0.8}^{*2})^\nu$	$(v_1^{*1})^\nu$	$(v_1^{*2})^\nu$
1	141.53	171.68	151.37	163.73	152.78	162.59	155.59	160.32
2	142.46	171.39	152.21	164.52	153.61	163.39	156.39	161.13
3	142.62	171.35	152.36	164.66	153.75	163.53	156.53	161.27
4	143.02	171.22	152.72	165.00	154.10	163.87	156.87	161.62

Table 6: End-points of α-cuts with respect to $\gamma_\mu = \{0, 0.6, 0.7, 1\}$ of membership functions of security levels for Player I.

#	$\widetilde{v(x^*)}^{\mu}$
1	(149.24, 151.86, 152.07, 155.59, 160.32, 162.78, 163.44, 165.59)
2	(150.08, 152.69, 152.91, 156.39, 161.13, 163.58, 164.24, 166.39)
3	(150.22, 152.82, 153.05, 156.53, 161.27, 163.71, 164.38, 166.35)
4	(150.59, 153.18, 153.42, 156.87, 161.62, 164.06, 164.26, 166.22)

Table 7: End-points of β-cuts with respect to $\gamma_\nu = \{0, 0.7, 0.8, 1\}$ of non-membership functions of security levels for Player I.

#	$\widetilde{v(x^*)}^{\nu}$
1	(141.53, 151.37, 152.78, 155.59, 160.32, 162.59, 163.73, 171.68)
2	(142.46, 152.21, 153.61, 156.39, 161.13, 163.39, 164.52, 171.39)
3	(142.62, 152.36, 153.75, 156.53, 161.27, 163.53, 164.66, 171.35)
4	(143.02, 152.72, 154.10, 156.87, 161.62, 163.87, 165.00, 171.22)

$$
\begin{aligned}
\max \quad & \Big((w_0^1)^\mu,\ (w_0^2)^\mu,\ (w_{0.6}^1)^\mu,\ (w_{0.6}^2)^\mu,\ (w_{0.7}^1)^\mu,\ (w_{0.7}^2)^\mu,\ (w_1^1)^\mu,\ (w_1^2)^\mu,\\
& (w_{0.8}^1)^\nu,\ (w_{0.8}^2)^\nu,\ (w_{0.7}^1)^\nu,\ (w_{0.7}^2)^\nu,\ (w_0^1)^\nu,\ (w_0^2)^\nu\Big)
\end{aligned}
$$

subject to

$$
\begin{aligned}
&175y_1 + 150y_2 \leqq (w_0^1)^\mu\,, && 80y_1 + 175y_2 \leqq (w_0^1)^\mu\,,\\
&190y_1 + 158y_2 \leqq (w_0^2)^\mu\,, && 100y_1 + 190y_2 \leqq (w_0^2)^\mu\,,\\
&177y_1 + 153.6y_2 \leqq (w_{0.6}^1)^\mu\,, && 84.3y_1 + 178y_2 \leqq (w_{0.6}^1)^\mu\,,\\
&188y_1 + 156.4y_2 \leqq (w_{0.6}^2)^\mu\,, && 97.43y_1 + 187y_2 \leqq (w_{0.6}^2)^\mu\,,\\
&177.75y_1 + 154.2y_2 \leqq (w_{0.7}^1)^\mu\,, && 83y_1 + 178y_2 \leqq (w_{0.7}^1)^\mu\,,\\
&187.25y_1 + 156.3y_2 \leqq (w_{0.7}^2)^\mu\,, && 97y_1 + 186.5y_2 \leqq (w_{0.7}^2)^\mu\,,\\
&180y_1 + 156y_2 \leqq (w_1^1)^\mu\,, && 90y_1 + 180y_2 \leqq (w_1^1)^\mu\,,\\
&185y_1 + 157y_2 \leqq (w_1^2)^\mu\,, && 94y_1 + 185y_2 \leqq (w_1^2)^\mu\,,\\
&178y_1 + 153.8y_2 \leqq (w_{0.8}^1)^\nu\,, && 85y_1 + 178y_2 \leqq (w_{0.8}^1)^\nu\,,\\
&187y_1 + 158.2y_2 \leqq (w_{0.8}^2)^\nu\,, && 97y_1 + 187y_2 \leqq (w_{0.8}^2)^\nu\,,\\
&177y_1 + 152.7y_2 \leqq (w_{0.7}^1)^\nu\,, && 82.5y_1 + 177y_2 \leqq (w_{0.7}^1)^\nu\,,\\
&188y_1 + 158.8y_2 \leqq (w_{0.7}^2)^\nu\,, && 98.5y_1 + 188y_2 \leqq (w_{0.7}^2)^\nu\,,\\
&170y_1 + 145y_2 \leqq (w_0^1)^\nu\,, && 65y_1 + 170y_2 \leqq (w_0^1)^\nu\,,
\end{aligned}
$$

$$195y_1 + 163y_2 \leqq (w_0^2)^\nu\,, \qquad 109y_1 + 195y_2 \leqq (w_0^2)^\nu\,,$$
$$y_1 + y_2 = 1\,, \qquad y_1, y_2 \geqq 0.$$

To solve the game for Player II, we need to solve the above multiobjective linear programming problem.

Table 8: POSS for Player II.

#	y_1^*	y_2^*
1	0.19	0.81
2	0.20	0.80
3	0.21	0.79
4	0.24	0.76

Table 9: Membership values of security levels for Player II.

#	$(w_0^{*1})^\mu$	$(w_0^{*2})^\mu$	$(w_{0.6}^{*1})^\mu$	$(w_{0.6}^{*2})^\mu$	$(w_{0.7}^{*1})^\mu$	$(w_{0.7}^{*2})^\mu$	$(w_1^{*1})^\mu$	$(w_1^{*2})^\mu$
1	155.21	171.25	158.48	168.34	159.11	167.85	161.25	166.04
2	155.26	171.05	158.53	168.14	159.16	167.66	161.05	165.84
3	155.38	171.42	158.65	168.51	159.07	168.02	161.42	166.21
4	155.57	170.59	158.83	168.68	159.02	168.19	161.59	166.39

Table 10: Non-membership values of security levels for Player II.

#	$(w_0^{*1})^\nu$	$(w_0^{*2})^\nu$	$(w_{0.7}^{*1})^\nu$	$(w_{0.7}^{*2})^\nu$	$(w_{0.8}^{*1})^\nu$	$(w_{0.8}^{*2})^\nu$	$(w_1^{*1})^\nu$	$(w_1^{*2})^\nu$
1	150.21	177.08	157.76	169.35	158.84	168.25	161.25	166.04
2	150.26	176.89	157.82	169.16	158.89	168.05	161.05	165.84
3	150.16	177.24	157.72	169.52	158.80	168.42	161.42	166.21
4	150.11	177.41	157.67	169.69	158.98	168.59	161.59	166.39

Table 11: End-points of α-cuts with respect to $\gamma_\mu = \{0, 0.6, 0.7, 1\}$ of membership functions of security levels for Player II.

#	$\widetilde{w(y^*)}^{\mu}$
1	(155.21, 158.48, 159.11, 161.25, 166.04, 167.85, 168.34, 171.25)
2	(155.26, 158.53, 159.16, 161.05, 165.84, 167.66, 168.14, 171.05)
3	(155.38, 158.65, 159.07, 161.42, 166.21, 168.02, 168.51, 171.42)
4	(155.57, 158.83, 159.02, 161.59, 166.39, 168.19, 168.68, 170.59)

Table 12: End-points of β-cuts with respect to $\gamma_\nu = \{0, 0.7, 0.8, 1\}$ of membership functions of security levels for Player II.

#	$\widetilde{w(y^*)}^{\nu}$
1	(150.21, 157.76, 158.84, 161.25, 166.04, 168.25, 169.35,177.08)
2	(150.26, 157.82, 158.89, 161.05, 165.84, 168.05, 169.16, 176.89)
3	(150.16, 157.72, 158.80, 161.42, 166.21, 168.42, 169.52,177.24)
4	(150.11, 157.67, 158.98, 161.59, 166.39, 168.59, 169.69,177.41)